List of contributions

Proceedings of the XIII Fall Workshop on
Geometry and Physics
Murcia, September 20–22, 2004
Publ. de la RSME, Vol. xx (2005), 2–12

Classical and quantum geometry of moduli spaces in three-dimensional gravity

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Abstract

We describe some results concerning the phase space of 3-dimensional Einstein gravity when space is a torus and with negative cosmological constant. The approach uses the holonomy matrices of flat $SL(2,\mathbb{R})$ connections on the torus to parametrise the geometry. After quantization, these matrices acquire non-commuting entries, in such a way that they satisfy q-commutation relations and exhibit interesting geometrical properties. In particular they lead to a quantization of the Goldman bracket.

Keywords: quantum geometry, moduli spaces, 3-dimensional spacetime, gravity, holonomy, Goldman bracket
2000 Mathematics Subject Classification: 83 C 45

1. Introduction

From the point of view of geometry, the theory of classical general relativity (see Fernando Barbero's lectures in this volume) is the study of Riemannian or semi-Riemannian geometries (depending on the choice of Euclidean or Lorentzian signature) which satisfy the Einstein equations. In 3-dimensional

spacetime these equations for the components $g_{\mu\nu}$ of the metric tensor are derived from the Einstein–Hilbert action

$$\int \sqrt{|g|}(R+\Lambda)d^3x \tag{1}$$

where integration is over the spacetime manifold, and we have included a cosmological constant Λ. In the first term of (1) the Ricci scalar, a contraction of the Riemann tensor, appears. This term may be written as follows:

$$\frac{1}{2}\int R^{\mu\sigma}{}_{\rho\nu}\epsilon_{\mu\sigma\alpha}\epsilon^{\rho\nu\beta}\delta^{\alpha}_{\beta}d^3x \tag{2}$$

where the usual summation convention over repeated indices is used and indices on the totally antisymmetric tensor $\epsilon_{\mu\nu\rho}$ are raised with the inverse metric tensor $g^{\mu\nu}$.

It is convenient to rewrite the action (1) in terms of orthonormal dreibeins or triads e^a. These are a local basis of 1-forms

$$e^a = e^a_\mu(x)\ dx^\mu, a = 1,2,3 \tag{3}$$

such that

$$g_{\mu\nu}dx^\mu \otimes dx^\nu = e^a \otimes e^b \eta_{ab} \tag{4}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1)_{ab}$. Then the action (1) takes the form

$$\int (R^{ab}\wedge e^c + \Lambda e^a \wedge e^b \wedge e^c)\epsilon_{abc} \tag{5}$$

where R^{ab} are the curvature 2-forms

$$R^{ab} = \frac{1}{2}R^{ab}{}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{6}$$

and $R^{ab}{}_{\mu\nu}$ is the Riemann tensor that appears in (2) contracted with the dreibein components (3).

In the dreibein formulation, there is an extra gauge symmetry of local Lorentz transformations $e^a \mapsto M^a{}_b e^b$ where $M \in SO(2,1)$ (local, since M depends on the point of spacetime). This extra freedom arises since one may simultaneously rotate the three fields e^a, whilst preserving the metric and the condition (4).

There is a striking similarity between the action in the form (5) and the Chern-Simons action for a connection A in a principal G-bundle, which has the structure

$$\int (F\wedge A + A\wedge A\wedge A).$$

Indeed, it was shown by Witten [1] that the action (5) may be interpreted as a Chern-Simons action for $G = SO(2,2)$, when $\Lambda < 0$ (and for $G = SO(3,1)$ when $\Lambda > 0$). The connection in the Chern-Simons theory is given in terms of the dreibein e^a and spin connection (or Ricci rotation coefficient) ω^{ab} by:

$$A = \frac{1}{2}\omega^{ab}M_{ab} + e^a M_{a4}, \tag{7}$$

where the indices a, b run from 1 to 3, and $\{M_{AB}\}_{A,B=1,\ldots,4}$ is a basis of the Lie algebra of $SO(2,2)$. Note that in this so-called first-order formalism, the dreibein e^a and spin connection ω^{ab} are independent fields.

We conclude this introduction with a short discussion of the relation between connections and holonomy. Given a connection on a principal G-bundle, a holonomy is an assignment of an element $H(\gamma)$ of G to each (based) loop γ on the manifold, obtained by lifting the loop into the total space of the bundle and comparing the starting and end points of the lifted loop in the fibre over the basepoint. Holonomy is, in a suitable sense, equivalent to the connection it is derived from. When the connection is flat, i.e. has zero fieldstrength F, the holonomy of γ only depends on γ up to homotopy. Thus an efficient way of describing flat connections is to specify a group morphism from the fundamental group of the manifold to the group G.

2. Equations of motion and the classical phase space

Consider the Chern-Simons action

$$\mathrm{tr}\int_{\Sigma\times\mathbb{R}} A\wedge dA + \frac{2}{3}A\wedge A\wedge A \tag{8}$$

on a spacetime of the form $\Sigma \times \mathbb{R}$, where Σ is a closed surface representing space and $\mathbb{R}$ represents time. The connection 1-form A may be written as

$$A = A_i dx^i + A_0 dx^0$$

where x^i, $i = 1, 2$ are coordinates on Σ, and $x^0 = t$ is the time coordinate. Imposing the gauge fixing condition

$$A_0 = 0$$

and the corresponding constraint

$$F_{ij}dx^i \wedge dx^j = 0$$

we see that the connections are flat. The action (8) now has the structure

$$\int A_2 \partial_0 A_1 + \dots$$

and therefore "A is its own conjugate momentum". The Poisson brackets for the components of A (see equation (7)) have the following form:

$$\left\{ {A_1}^a(x), {A_j}^b(y) \right\} = \delta^{ab} \epsilon_{ij} \delta^2(x-y) \quad \epsilon_{12} = 1. \tag{9}$$

We now choose the space manifold to be the torus $\mathbb{T}^2$, and since the group $SO(2,2)$ is isomorphic, up to a discrete identification, to the product $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, we restrict ourselves to studying the phase (moduli) space of flat $SL(2,\mathbb{R})$ connections on the torus $\mathbb{T}^2$, modulo gauge transformations. Note that this is in principle a complicated space to describe, being an infinite-dimensional space divided by an infinite-dimensional group, but in the holonomy picture there is a very simple finite-dimensional description.

Since $\pi_1(\mathbb{T}^2) = \left\langle \gamma_1, \gamma_2 | \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} = 1 \right\rangle$ where γ_1 and γ_2 are a pair of generating cycles, a holonomy

$$H : \pi_1(\mathbb{T}^2) \to SL(2,\mathbb{R})$$

is given by $U_1 := H(\gamma_1)$ and $U_2 := H(\gamma_2)$, since this determines H on any other homotopy class of loops. The phase space P is then

$$P = \{(U_1, U_2) | U_1 U_2 = U_2 U_1\} / \sim$$

where $\sim$ denotes the remaining gauge freedom, namely

$$(U_1, U_2) \sim (S^{-1} U_1 S, S^{-1} U_2 S)$$

for any $S \in SL(2,\mathbb{R})$.

For a single matrix $U \in SL(2,\mathbb{R})$ there are four possibilities for how U can be conjugated into a standard form:

A) U has 2 real eigenvalues:

$$S^{-1} U S = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

B) U has 1 real eigenvalue with an eigenspace of dimension 2:

$$S^{-1} U S = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

C) U has 1 real eigenvalue with an eigenspace of dimension 1:

$$S^{-1}US = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$$

D) U has no real eigenvalues:

$$S^{-1}US = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A similar analysis for a pair of commuting $SL(2,\mathbb{R})$ matrices led in [2] to an explicit parametrization of the classical phase space P. Its structure resembles that of a cell complex with, for instance 2-dimensional cells consisting of pairs of diagonal matrices, or pairs of rotation matrices. However there are also 1-dimensional cells which consist of e.g. pairs of non-diagonalisable matrices of the form:

$$U_1 = \begin{pmatrix} 1 & \cos\alpha \\ 0 & 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & \sin\alpha \\ 0 & 1 \end{pmatrix}, \; 0 < \alpha < \frac{\pi}{2}. \tag{10}$$

For further details and depictions of P see [2].

3. Quantization via quantum matrices

The Poisson brackets (9) are for non-gauge-invariant variables so it is convenient to change to gauge-invariant variables, and an obvious choice are the traced holonomies

$$T(\gamma) = \frac{1}{2} \text{ tr } H(\gamma)$$

which are gauge-invariant due to the conjugation invariance of the trace. The holonomy is sometimes written as a path-ordered exponential, or Chen integral,

$$H(\gamma) = \mathcal{P} \exp \int_\gamma A$$

and from equation (9) the Poisson brackets between the $T(\gamma)$ are only non-vanishing if the loops intersect transversally. From trace identities for 2×2 matrices it is enough to consider the following three variables:

$$T_1 := T(\gamma_1) \quad T_2 := T(\gamma_2) \quad T_3 := T(\gamma_1\gamma_2)$$

(which are not independent since they satisfy the identity $T_1^2 + T_2^2 + T_3^2 - 2T_1T_2T_3 = 1$). Their Poisson bracket relations are [3]

$$\{T_i, T_j\} = \epsilon_{ij}{}^k T_k + T_i T_j, \quad i,j,k = 1,2,3. \tag{11}$$

(here we have rescaled the variables compared to [3] to absorb the coupling constants).

The first term on the right-hand-side of equation (11) has a geometric interpretation in terms of rerouted loops: e.g. for $i = 1$, $j = 2$ the two cycles γ_1 and γ_2 intersect transversally at one point, and from homotopy invariance of the holonomy T_3 is the traced holonomy corresponding to the loop $\gamma_1 S \gamma_2$ obtained by starting at the basepoint, following γ_1 to the intersection point S, rerouting along the loop γ_2 back to the intersection point, and finally continuing again along γ_1 back to the basepoint. We will see more of these rerouted loops shortly.

We observe that by parametrising the variables as follows:

$$T_1 = \cosh r_1 \quad T_2 = \cosh r_2 \quad T_3 = \cosh(r_1 + r_2)$$

equation (11) is solved by setting:

$$\{r_1, r_2\} = 1.$$

On quantization, replacing T_i, r_j by operators $\hat{T}_i$, $\hat{r}_j$ respectively implies the corresponding commutation relation:

$$[\hat{r}_1, \hat{r}_2] = i\hbar. \tag{12}$$

The operators $\hat{T}_i$ satisfy a q-deformed ($q = e^{i\hbar}$) cubic relation, which can be interpreted in terms of a quantum Casimir operator for the quantum group $SU(2)_q$ - see [4].

We note that e.g.

$$T_1 = \frac{1}{2} \text{ tr } U_1 = \cosh r_1 = \frac{1}{2}(e^{r_1} + e^{-r_1})$$

so that by introducing the *quantum* matrices

$$\hat{U}_i = \begin{pmatrix} e^{\hat{r}_i} & 0 \\ 0 & e^{-\hat{r}_i} \end{pmatrix} \; i = 1, 2 \tag{13}$$

we have the analogous relation between $\hat{T}_i$ and $\hat{U}_i$, namely

$$\hat{T}_i = \frac{1}{2} \text{ tr } \hat{U}_i \quad i = 1, 2.$$

We also notice that these quantum matrices satisfy the following fundamental relation:

$$\hat{U}_1 \hat{U}_2 = q \; \hat{U}_2 \hat{U}_1, \tag{14}$$

where we are using matrix multiplication of operator-valued matrices (the usual algebraic rule, but paying strict attention to the order of the symbols). For example, the relation

$$e^{\hat{r}_1} e^{\hat{r}_2} = q\ e^{\hat{r}_2} e^{\hat{r}_1}$$

follows from the commutation relation (12) between the operators $\hat{r}_i$.

The cubic constraint satisfied by the quantum variables $\hat{T}_i$ is rather complicated, so instead we work with the quantum holonomy matrices $\hat{U}_i$ themselves rather than with the trace functions $\hat{T}_i$. It is important to note that even though the quantum matrices $\hat{U}_i$ are not gauge-invariant, i.e.

$$\hat{U}_i \neq S^{-1} \hat{U}_i S$$

for general S, the fundamental equation (14) is gauge-covariant, and is also covariant under the modular symmetry of the theory, i.e. the group of large diffeomorphisms of the torus - see [5]. Thus our idea is to substitute invariant variables obeying complicated equations by non-invariant matrix variables satisfying natural q-commutation relations like the fundamental relation (14). Certainly for the case of diagonal matrices these two viewpoints are entirely equivalent.

We have also studied, in [5], what happens when one imposes the fundamental equation for a pair of upper-triangular quantum matrices, which should correspond, in some sense, to the quantization of the 1-dimensional upper-triangular cell of the classical phase space mentioned in section 2.

If one parametrizes the quantum matrices $\hat{U}_i$ as follows:

$$\hat{U}_i = \begin{pmatrix} \hat{\alpha}_i & \hat{\beta}_i \\ 0 & \hat{\alpha}_i^{-1} \end{pmatrix}, \tag{15}$$

where the $\hat{\alpha}_i$, $\hat{\beta}_i$ are operators to be determined, a solution to equation (14) is given by:

$$\begin{aligned} \hat{\alpha}_1 \psi(b) &= \exp \frac{d}{db} \psi(b) \\ \hat{\alpha}_2 \psi(b) &= \exp i\hbar b\, \psi(b) \\ \hat{\beta}_i \psi(b) &= \hat{\alpha}_i \psi(-b) \end{aligned} \tag{16}$$

Note the change of sign in the argument of ψ in the last of equations (16). It can be checked, from (16) that

$$\hat{\alpha}_1 \hat{\alpha}_2 = q\ \hat{\alpha}_2 \hat{\alpha}_1$$

as required, but we also get an *internal* commutation relation

$$\hat{\alpha}_1 \hat{\beta}_1 = \hat{\beta}_1 \hat{\alpha}_1^{-1}$$

for the elements of $\hat{U}_1$ and similarly for $\hat{U}_2$, which curiously does not involve the quantum parameter $\hbar$.

Note that it is impossible to find solutions to (14) with $\hat{\alpha}_i = \mathbb{I}$, the unit operator, by naive analogy with equation (10), since $\hat{\beta}_1 + \hat{\beta}_2 \neq q(\hat{\beta}_1 + \hat{\beta}_2)$. Thus in terms of the number of quantum parameters, this upper-triangular sector would appear to be as substantial as the triangular sector, unlike the classical case.

Finally we remark that in [6], we studied equations like (15) from an algebraic point of view, and found that their solutions have several interesting properties analogous to quantum groups.

4. Reroutings and the quantized Goldman bracket

Here we briefly describe our most recent work - for a full treatment see [7]. In section 3 we only considered the quantum matrices assigned to γ_1 and γ_2, so it is natural to try and understand how to assign quantum matrices to other loops, and to study the relationships between them. A useful way of doing this, proposed in [8], is to introduce a constant quantum connection

$$\hat{A} = (\hat{r}_1 dx + \hat{r}_2 dy) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where constant means that the $\hat{r}_i$ do not depend on the spatial coordinates x, y of the torus. Then the assignment of a quantum matrix to any loop is given by the holonomy of this connection along the loop:

$$\gamma \mapsto \hat{U}_\gamma = \exp \int_\gamma \hat{A}. \tag{17}$$

It can easily be seen that (17) reproduces the quantum matrices $\hat{U}_i$ of equation (13), if γ_1 is the loop with y coordinate constant and x running from 0 to 1, and γ_2 is the loop with x constant and y running from 0 to 1.

It is convenient to identify the torus $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ consists of points with integer x and y coordinates. We consider all loops on the torus represented by piecewise-linear (PL) paths between integer points on the (x, y) plane, and work with this description, keeping in mind that paths represent loops. In particular the integer paths denoted (m, n) are straight paths between $(0, 0)$ and (m, n) with m, n integers. Thus for example we assign to the integer path $(2, 1)$ the quantum matrix

$$\hat{U}_{(2,1)} = \begin{pmatrix} e^{2\hat{r}_1 + \hat{r}_2} & 0 \\ 0 & e^{-2\hat{r}_1 - \hat{r}_2} \end{pmatrix}.$$

Consider two homotopic loops γ_1 and γ_2 corresponding to PL paths both starting at $(0,0)$ and ending at the same integer point in the plane. It was shown in [7] that there is the following relationship between the respective quantum matrices:

$$\hat{U}_{\gamma_1} = q^{S(\gamma_1,\gamma_2)}\hat{U}_{\gamma_2}, \tag{18}$$

where $S(\gamma_1,\gamma_2)$ denotes the signed area enclosed between the paths γ_1 and γ_2. For example, the exponent (the number 1) of q in the fundamental relation (14) is the signed area between two paths around the perimeter of the unit square, starting at $(0,0)$ and ending at $(1,1)$, the first via $(1,0)$ and the second via $(0,1)$.

The traces of these quantum matrices also exhibit commutation relations with interesting properties. Let

$$\hat{T}(m,n) := \operatorname{tr} \hat{U}_{(m,n)}.$$

(note we have dropped the factor $\frac{1}{2}$ for easier comparison with the Goldman result below). It was shown in [7] that the following commutation relation holds:

$$[\hat{T}(m,n),\hat{T}(s,t)] = (q^{\frac{mt-ns}{2}} - q^{-\frac{mt-ns}{2}})\left(\hat{T}(m+s,n+t) - \hat{T}(m-s,n-t)\right) \tag{19}$$

There are some surprising geometric aspects to equation (19). The number $mt-ns$ appearing in the exponents is the signed area of the parallelogram spanned by the vectors (m,n) and (s,t). The same expression equals the suitably-defined total intersection number (including and counting multiplicities) of the two loops represented by the paths (m,n) and (s,t). Equation (19) can, in fact, be viewed as a quantization of a bracket due to Goldman [9] for the loops corresponding to such integer paths. This bracket is a Poisson bracket for the functions $T(\gamma) = \operatorname{tr} U_\gamma$ given by:

$$\{T(\gamma_1),T(\gamma_2)\} = \sum_{S\in\gamma_1\sharp\gamma_2} \epsilon(\gamma_1,\gamma_2,S)(T(\gamma_1 S\gamma_2) - T(\gamma_1 S\gamma_2^{-1})) \tag{20}$$

where $\gamma_1\sharp\gamma_2$ denotes the set of transversal intersection points of γ_1 and γ_2 and $\epsilon(\gamma_1,\gamma_2,S)$ is their intersection index for the intersection point S. In equation (20) $\gamma_1 S\gamma_2$ and $\gamma_1 S\gamma_2^{-1}$ denote the loops which follow γ_1 and are rerouted along γ_2, or its inverse, at the intersection point S as described previously. For the integer loops considered here, all the rerouted loops $\gamma_1 S\gamma_2$ are homotopic to the integer loop $(m+s,n+t)$, with an analogous statement for the loops $\gamma_1 S\gamma_2^{-1}$. It follows that the classical Goldman bracket (20) can be written as

$$\{T(m,n),T(s,t)\} = (mt-ns)(T(m+s,n+t) - T(m-s,n-t)).$$

Therefore the first factor on the right hand side of (19) may be thought of as a quantum total intersection number for the loops (m, n) and (s, t).

We remark that in [7] we also derived a different form of (19) where each rerouted loop appears separately. The different terms are related by the same area phases as in (18). In these proofs we used a classical geometric result [10] namely Pick's Theorem (1899), which expresses the area $A(P)$ of a lattice polygon P with vertices at integer lattice points of the plane in terms of the number of interior lattice points $I(P)$ and the number of boundary lattice points $B(P)$ as follows:

$$A(P) = I(P) + \frac{B(P)}{2} - 1.$$

Full details are given in [7].

Acknowledgments

This work was supported by the Istituto Nazionale di Fisica Nucleare (INFN) of Italy, Iniziativa Specifica FI41, the Italian Ministero dell'Università e della Ricerca Scientifica e Tecnologica (MIUR), and by the programme *Programa Operacional "Ciência, Tecnologia, Inovação"* (POCTI) of the *Fundação para a Ciência e a Tecnologia* (FCT), cofinanced by the European Community fund FEDER.

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